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Prescribed Curvature and Singularities of Conformal Metrics on Riemann Surfaces

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0. INTRODUCTION

Suppose M is a compact Riemann surface with Euler characteristic $\chi(M)$. Let g be a compatible metric, and K denote its Gaussian curvature. Suppose we are given a function $\tilde{K} \in C^\infty(M)$; the problem of prescribed curvature is to find a metric $\tilde{g}$ which is (pointwise) conformal to g and has $\tilde{K}$ as its curvature (cf. [8]). If we write $\tilde{g} = e^{2u}g$, this is equivalent to solving the familiar equation

$$\Delta u + \tilde{K}e^{2u} = K. \tag{$*$}$$

(Of course we could consider $\tilde{K}$ in a C^α or L^p space, but we generally work in the smooth category.)

But now suppose we are also given a finite set of points, $\Gamma = \{p_1, ..., p_n\} \subset M$, and we let $\hat{M} = M \backslash \Gamma$. Then for $\tilde{K} \in C^\infty(\hat{M})$, the problem of prescribed curvature is to find a metric $\tilde{g}$ on $\hat{M}$ which is conformal to g and has $\tilde{K}$ as its curvature. Again we must be able to solve $(*)$, but the question naturally arises as to what happens as $r_i = \mathrm{dist}(x, p_i) \to 0$.

In [11], this problem was considered when $\tilde{K} \equiv -1$, and it was found that we may also prescribe singularities or degeneracies at the p_i: we may solve $(*)$ with u satisfying

$$u = \alpha_i \log r_i + c + o(1) \qquad \text{as} \quad r_i \to 0 \tag{0.1}$$

provided (i) $-1 < \alpha_i < \infty$, and (ii) $\chi(M) + \sum_i \alpha_i < 0$. (Here and throughout this paper we write $f = c + o(1)$ to mean that for some constant c we have $\lim_{r_i \to 0}(f - c) = 0$; note that (0.1) does not require c to have the same value at each p_i.)

In fact, it was later brought to the author's attention by D. Minda that this problem was first treated by Picard [13], and later generalized by

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M. Heins [5] to allow $\alpha_i = -1$ in condition (i) provided the asymptotic (0.1) is replaced by

$$u = \log \frac{1}{r_i} - \log \left(\log \frac{1}{r_i} \right) + c + o(1) \qquad \text{as} \quad r_i \to 0. \tag{0.2}$$

Moreover, in [14], M. Troyanov (independently from [11]) studied the prescribed curvature problem on $\hat{M}$ with solutions of the form (0.1). He considered various cases of sign conditions on $\tilde{K}$ (analogous to the compact case in [8]). In particular, his results for $\tilde{K} < 0$ on M generalize those of [11].

In this paper we study the prescribed curvature problem on $\hat{M}$ in the case where $\tilde{K} \leqslant 0$ on $\hat{M}$ with the behavior at p_i,

$$\tilde{K} = O(r_i^{l_i}) \qquad \text{as} \quad r_i \to 0, \tag{0.3}$$

where $O(r^l)$ means bounded by Cr^l, or more restrictively

$$\tilde{K} \approx -r^{l_i} \qquad \text{as} \quad r_i \to 0, \tag{0.4}$$

where we define $\tilde{K} \approx -r^l$ to mean $-C_1 r^l \leqslant \tilde{K} \leqslant -C_2 r^l$ for some positive constants $C_1 > C_2$. Note that we need not *a priori* require $l_i \geqslant 0$, i.e., $\tilde{K}$ may be unbounded, but we shall see that more solutions exist for larger l_i. In Theorem I we find that, assuming (0.3), solutions of the form (0.1) always exist provided the α_i satisfy a local integrability condition (generalizing (i) above and guaranteeing finite total curvature $\tilde{k}$) and a global integrability condition ((ii) above, guaranteeing $\tilde{k} < 0$). In Theorem II we find that (0.4) implies the existence of additional solutions (which also have finite negative total curvature $\tilde{k}$); moreover, in this case these are *all* of the solutions of (*). (In fact, in Appendix B of this paper we discuss the local behavior of solutions near p_i: this is of independent interest since it does not use the global assumption $\tilde{K} \leqslant 0$ on $\hat{M}$.)

The results of this paper generalize earlier work on $\mathbb{R}^2$ (cf. [12, 9, 2, 3]) and on $S^1 \times \mathbb{R}^1$ (cf. [15]), by considering these noncompact surfaces as the sphere S^2 punctured by one or two points. We shall discuss these comparisons in further detail in Section 2 below.

After completing this paper, I received a recent preprint from D. Hulin and M. Troyanov [7] which makes a more comprehensive study of the problem of prescribing curvature on open Riemann surfaces. In particular, they consider prescribing a nonpositive curvature function for a singular conformal metric on a Riemann surface which itself may be noncompact, provided it has hyperbolic ends. In the special case of a compact surface, the results they obtain essentially coincide with the existence results in this paper (although there are minor differences, such as the hypothesis $-2\alpha_i <$

l_i+2 of Theorem I below is replaced in [7] by the condition $r^{2\alpha_i}\tilde{K} \in L^p$ for some $p>1$ in a neighborhood of p_i). However, the results in [7] do *not* include the local and global behavior of *all* solutions of $(*)$ which are obtained in this paper when (0.4) holds.

1. Statements and Proofs of Theorems

Throughout this section we assume, as in the introduction, that M is a compact Riemann surface with Euler characteristic $\chi(M)$ and g is a compatible metric with Gaussian curvature K. Our first theorem generalizes the result of [11] which corresponds to the special case $l_i=0$.

Theorem I. *Let* $\Gamma = \{p_1, \ldots, p_n\} \subset M$, *and* $\hat{M} = M \setminus \Gamma$. *Suppose* $\tilde{K} \in C^\infty(\hat{M})$ *such that* $\tilde{K} \leqslant 0$ *on* $\hat{M}$, $\tilde{K}(x)<0$ *for some* $x \in \hat{M}$, *and* $\tilde{K} = O(r_i^{l_i})$ *as* $r_i \to 0$ *where* l_i *satisfies* $\sum_i l_i > 2(\chi(M)-n) = 2\chi(\hat{M})$. *Then for every* $\alpha_i > -(l_i+2)/2$ *satisfying* $\sum_i \alpha_i < -\chi(M)$ *there exists a unique solution* u *of* $(*)$ *satisfying*

$$u = \alpha_i \log r_i + c + o(1) \qquad \text{as} \quad r_i \to 0. \tag{1.1}$$

Moreover, the resulting conformal metric $\tilde{g} = e^{2u}g$ *has total curvature*

$$\tilde{k} = 2\pi\left(\chi(M) + \sum_i \alpha_i\right). \tag{1.2}$$

Proof. Theorem I was proved in [11] for the special case $\tilde{K} \equiv -1$ (i.e., $l_i=0$); the proof generalizes easily, but for the sake of the exposition of this paper (and due to a couple of minor errors in the proof in [11]) we briefly present it here.

First, since Riemann surfaces are locally conformally flat, we may assume g is Euclidean in a neighborhood of each point p_i. Analytically, this may easily be achieved as follows: otherwise, let $K_0 \in C^\infty(M)$ satisfy (i) $K_0(x)=0$ for $0<\operatorname{dist}_g(x, p_i)<\varepsilon$ and (ii) $\int K_0\, dA = \int K\, dA$; then solve $\Delta u_0 = K - K_0$ for $u_0 \in C^\infty(M)$. The metric $g_0 = e^{2u_0}g$ has curvature $K_0 e^{-2u_0}$, hence is Euclidean near each p_i, and the proof may proceed using g_0 in place of g.

Henceforth we assume for notational convenience that $n=1$ although all steps generalize immediately.

Given $\alpha \in (-(l+2)/2, -\chi(M))$, pick $u_1 \in C^\infty(\hat{M})$ such that $u_1 = \alpha \log r$ for $0<r<\varepsilon$ (where g is Euclidean), so $\int \Delta u_1\, dA = -2\pi\alpha$. Next, let $u_0 \in C^\infty(M)$ solve

$$\Delta u_0 = K - \Delta u_1 - \frac{2\pi(\chi(M)+\alpha)}{\operatorname{area}(M)} \tag{1.3}$$

(note the right hand side has integral zero), and write $u = u_1 + u_0 + w$. Thus, in order to solve $(*)$ we must solve

$$\Delta w + he^{2w} = c, \tag{1.4}$$

where $h(x) = \tilde{K}e^{2(u_1+u_0)} \leqslant 0$, $h = O(r^{l+2\alpha})$, and $c = 2\pi(\chi(M)+\alpha)/\text{area}(M) < 0$.

Note that $h \in L^p(M)$ for some $p > 1$ (since $l + 2\alpha > -2$). This enables us to solve (1.4) for w bounded on M using upper and lower solutions, similar to [8] (for further details, see [11]). Thus $w \in H_2^p(M) \subset C^\beta(M)$ for $0 < \beta < 2 - 2/p$, so (1.1) holds.

To establish uniqueness, suppose u and u' are two solutions of $(*)$ satisfying (1.1). Then $u = u_1 + v$ and $u' = u_1 + v'$, where $v, v' \in C(M)$. Thus $U = u - u' = v - v' \in C(M) \cap C^\infty(\hat{M})$ satisfies

$$\left.\begin{aligned} &\Delta U + c(x)U = 0 \qquad \text{in } \hat{M} \\ &\text{where} \quad c(x) = \tilde{K}e^{2u_1}\frac{(e^{2v} - e^{2v'})}{v - v'} \leqslant 0 \end{aligned}\right\}. \tag{1.5}$$

Thus $U^+ = \max(U, 0)$ is subharmonic on $\hat{M}$ and continuous on M, so subharmonic on M. But M is compact, so $U^+ = \text{const}$. Since $\tilde{K}$ is nontrivial, we must have $U^+ \equiv 0$, i.e., $U \leqslant 0$. Similarly, $U \geqslant 0$, so we must have $U \equiv 0$, i.e., the solution is unique.

Finally, to verify the formula (1.2) we note that $w \in H_2^p(M)$ implies $\int \Delta w \, dA = 0$, and hence

$$\int \tilde{K}\, d\tilde{A} = \int \tilde{K}e^{2u}\, dA = \int K\, dA - \int \Delta(u_1 + u_0 + w)\, dA = 2\pi(\chi(M) + \alpha)$$

as desired.

Our next theorem shows that under the stronger assumption (0.4) on $\tilde{K}$ near p_i, we can assert the existence of additional solutions corresponding to the choice $\alpha_i = -(l_i + 2)/2$.

THEOREM II. *Let $\Gamma = \{p_1, ..., p_n\} \subset M$, and $\hat{M} = M \backslash \Gamma$. Suppose $\tilde{K} \in C^\infty(\hat{M})$ such that $\tilde{K} \leqslant 0$ on $\hat{M}$ and $\tilde{K} \approx -r^{l_i}$ as $r_i \to 0$ where l_i satisfies $\sum_i l_i > 2(\chi(M) - n) = 2\chi(\hat{M})$. Then for every $\alpha_i \geqslant -(l_i + 2)/2$ satisfying $\sum_i \alpha_i < -\chi(M)$ there exists a unique solution u of $(*)$ satisfying*

$$u = \alpha_i \log r_i + c + o(1) \qquad \text{as} \quad r_i \to 0 \text{ if } \alpha_i > -\frac{l_i + 2}{2} \tag{1.6}$$

$$u = \frac{l_i + 2}{2}\log\frac{1}{r_i} - \log\left(\log\frac{1}{r_i}\right) + O(1) \qquad \text{as } r_i \to 0 \text{ if } \alpha_i = -\frac{l_i + 2}{2}. \tag{1.7}$$

Moreover, (i) *the resulting conformal metric* $\tilde{g} = e^{2u}g$ *has total curvature*

$$\tilde{k} = 2\pi\left(\chi(M) + \sum_i \alpha_i\right), \tag{1.8}$$

and (ii) *the solutions with asymptotics* (1.6) *and* (1.7) *are "complete" in that* every *solution of* $(*)$ *agrees with one of these.*

Proof. First pick a neighborhood N_i of each p_i on which $\tilde{K} < 0$, and then $K_0 \in C^\infty(M)$ such that $K_0 = 0$ near p_i, $\operatorname{supp} K_0 \subset \bigcup \{N_i\}$, and $\int K_0\, dA = \int K\, dA$. As in Theorem I, we may assume $n = 1$, $K = 0$ for $0 < r < \varepsilon$, and $\tilde{K} < 0$ on supp K. Also note that we have $-\tilde{C}_1 r^l \leqslant \tilde{K} \leqslant -\tilde{C}_2 r^l$ for $0 < r < \varepsilon$.

By Theorem I, we are concerned with the case $\alpha = -(l+2)/2$. The next step is to construct a "barrier function" near p_1. Let us define

$$u_+ = \frac{l+2}{2}\log\frac{1}{r} - \log\left(\log\frac{\varepsilon}{r}\right) + b \qquad \text{for} \quad 0 < r < \varepsilon, \tag{1.9}$$

where we choose b below. Note that $u_+ \to \infty$ as $r \to 0$ or $r \to \varepsilon$. Since g is Euclidean, for $0 < r < \varepsilon$ we may compute

$$\Delta u_+ = \frac{1}{r^2(\log(\varepsilon/r))^2}$$

so

$$\Delta u_+ + \tilde{K}e^{2u_+} \leqslant \frac{1}{r^2(\log(\varepsilon/r))^2} - \tilde{C}_2 \frac{e^{2b}}{r^2(\log(\varepsilon/r))^2} \leqslant 0$$

if b is sufficiently large. Thus u_+ is an upper solution on its domain (cf. Appendix A).

Now let α_k denote a sequence $\alpha_k \downarrow -(l+2)/2$ and let u_k denote the corresponding solutions satisfying

$$u_k = \alpha_k \log r + c + o(1) \qquad \text{as} \quad r \to 0 \tag{1.10}$$

(provided by Theorem I). By the maximum principle (cf. Lemma 1 in Appendix A), $u_k < u_{k+1}$ on $\hat{M}$. Moreover, $\{u_k\}$ is uniformly bounded above since, for $0 < r < \varepsilon$ we have $u_k < u_+$ again by the maximum principle (Lemma 1), and for $r > \varepsilon/2$ we have an apriori bound: if u_k has a relative maximum at x_0, then we may assume $\tilde{K}(x_0) < 0$ (since u_k is subharmonic on the set where $\tilde{K} = 0$) and hence $e^{2u_k(x_0)} \leqslant K(x_0)/\tilde{K}(x_0)$. Thus u_k converges, uniformly on compact subsets of $\hat{M}$, to a solution U of $(*)$ on $\hat{M}$. Clearly $U \leqslant u_+$ on $0 < r < \varepsilon$; but $U < u_+$ near $r = \varepsilon$, so applying Lemma 1 on any $\delta < r < \varepsilon - \delta$ shows that $U < u_+$ on $0 < r < \varepsilon$.

Clearly, U is not of the form (1.6) near $r=0$; by our knowledge of *all* solutions near $r=0$ (cf. Appendix B) we conclude that

$$U=\frac{l+2}{2}\log\frac{1}{r}-\log\left(\log\frac{1}{r}\right)+O(1). \tag{1.11}$$

To establish the formula (1.8), note that $\tilde{K}e^{2u_k}\downarrow\tilde{K}e^{2U}$ and so by the Monotone Convergence Theorem,

$$\int \tilde{K}e^{2u_k}\,dA=2\pi(\chi(M)+\alpha_k)\to\int \tilde{K}e^{2U}\,dA,$$

with the last integral being finite since $\tilde{K}e^{2U}=O(r^{-2}(\log(1/r))^{-2})\in L^1(M)$. Thus $\tilde{k}=\int \tilde{K}e^{2U}\,dA=2\pi(\chi(M)-(l+2)/2)$ as to be shown.

To establish uniqueness of the solution U satisfying (1.11), let us first assume that $\hat{M}$ is a parabolic Riemann surface, i.e., admits no bounded subharmonic functions other than constants. If U_1 and U_2 both satisfy (1.11), then $W=U_1-U_2$ is bounded and W^2 is subharmonic since (following the proof of Theorem 4.1 in [3])

$$\begin{aligned}\Delta(W^2)&=2W\,\Delta W+2\,|\nabla W|^2\geqslant 2W\,\Delta W\\&=-2\tilde{K}(U_1-U_2)^2\left[\frac{e^{2U_1}-e^{2U_2}}{U_1-U_2}\right]\geqslant 0.\end{aligned}$$

Hence W must be a constant, which must be zero since $\tilde{K}$ is not identically zero, so the solution is unique.

To establish uniqueness when $\hat{M}$ is hyperbolic, let $\hat{g}$ denote the hyperbolic metric on $\hat{M}$: $\hat{g}$ is complete with curvature $\hat{K}\equiv -1$ and $\hat{g}=e^{2\hat{u}}g$ where (by the Local Schwarz Lemma)

$$\hat{u}=\log\frac{1}{r}-\log\log\frac{1}{r}+O(1).$$

If U_1 and U_2 both satisfy (1.11), then $W=U_1-U_2$ is bounded and

$$\Delta_{\hat{g}}W=e^{-2\hat{u}}\Delta_g W=e^{-2\hat{u}}(-\tilde{K})[e^{2U_1}-e^{2U_2}] \tag{1.12}$$

or

$$\Delta_{\hat{g}}W+\frac{e^{-2\hat{u}}\tilde{K}[e^{2U_1}-e^{2U_2}]}{U_1-U_2}\,W=0.$$

If W attains a nonpositive minimum or nonnegative maximum on $\hat{M}$, then W is a constant, which must be zero since $\tilde{K}$ is not identically zero. On the other hand, by the Generalized Maximum Principle (cf. [16]), there exist points $\{q_k\} \subset \hat{M}$ such that

$$\lim_{k \to \infty} W(q_k) = \sup W, \qquad \limsup_{k \to \infty} \Delta_{\hat{g}} W(q_k) \leqslant 0.$$

We must have q_k eventually outside every compact subset of $\hat{M}$. But using (1.12) and the known behavior of $\hat{u}$, U_i, and $\tilde{K}$ as $r \to 0$, we must have

$$\limsup [e^{2V_1(q_k)} - e^{2V_2(q_k)}] \leqslant 0,$$

where $U_j = (l+2)/2 \log(1/r) - \log(\log(1/r)) + V_j$. Thus $W \leqslant 0$. Similarly, we prove $W \geqslant 0$, and hence obtain uniqueness.

Finally, the "completeness" of these solutions follows from the description of all local solutions in Appendix B together with the uniqueness statement in Theorem I. This completes the proof of Theorem II.

Remark. A perusal of the proofs shows that we can combine the behavior (0.3) at some points and (0.4) at others. For example, if $\Gamma = \{p_1, p_2\}$ and $\tilde{K} = O(r_1^{l_1})$ as $r_1 \to 0$ but $\tilde{K} \approx -r_2^{l_2}$ as $r_2 \to 0$, then there is a unique solution of ($*$) satisfying (1.1) for $i = 1$ and (1.6) or (1.7) for $i = 2$.

2. Further Discussion of the Results

For the sake of discussion, let us mostly consider $n = 1$ so that $\hat{M} = M \setminus \{\text{pt.}\}$. Note that the metric $\tilde{g} = e^{2u} g$ where u satisfies (1.6) or (1.7) is complete if and only if $\alpha \leqslant -1$. Thus, the existence of a complete conformal metric with curvature $\tilde{K}$ in Theorem I requires $l > 0$ and in Theorem II requires $l \geqslant 0$. In particular, if $\tilde{K} \equiv -1$, then with $l = 0$ the condition $l + 2 > 2\chi(M)$ is equivalent to $\chi(\hat{M}) = \chi(M) - 1 < 0$, so Theorem II implies that $(\hat{M}, g)$ is hyperbolic, i.e., admits a complete conformal metric with constant negative curvature, provided $\chi(\hat{M}) < 0$, a well-known fact from uniformization theory (but unattainable from Theorem I).

We can also compare Theorem II with the Cohn–Vossen inequality which states that if $(\hat{M}, \tilde{g})$ is complete then the total curvature $\tilde{k}$ satisfies $\tilde{k} \leqslant 2\pi\chi(\hat{M})$; in Theorem II we see that the completeness of the metric is equivalent to the validity of the Cohn–Vossen inequality. (This was observed by P. Aviles [1] when $\hat{M} = \mathbb{R}^2$.)

We can interpret both Theorems I and II as stating that for the noncompact manifold $\hat{M}$ we can conformally deform our metric g to obtain a metric $\tilde{g}$ with the prescribed curvature function $\tilde{K}$ *and* a prescribed value for the total curvature $\tilde{k}$ in a certain range $[2\pi(\chi(M) - (l+2)/2) < \tilde{k} < 0$ in Theorem I, with the left end-point included in Theorem II]. This was observed in [10] when $\hat{M} = \mathbb{R}^2$.

In the special case $M=S^2$, we have $\chi(M)=2$ so, in order to obtain $l+2>2\chi(M)$, we require $l>2$, a condition to be found in various papers on the $\mathbb{R}^2$ case. For example, the Theorem in [9] is seen to be a special case of Theorem I above, and Theorem II in [3] is a special case of Theorem II above, although Cheng and Ni obtain only the asymptotic (1.6) with $O(1)$ in place of $c+o(1)$, and do not compute the total curvature of $\tilde{g}$. In fact, Cheng and Ni obtain the existence of the solution U as the "maximal solution" under more general conditions on $\tilde{K}$, but then obtain the asymptotic (1.7) under the additional assumption (0.4).

If $M=S^2$ and $\Gamma=\{p_1, p_2\}$, then $\hat{M}$ is conformally the flat cylinder and we may compare Theorems I and II with Theorem 1a in [15] (where it was assumed that $l_1=l_2=l$). The conditions $l_1+l_2>2\chi(M)$, $\alpha_i>-(l_i+2)/2$ and $\alpha_1+\alpha_2<-\chi(M)$ appear in [15] respectively as $l>0$, $\max\{\tilde{\alpha}_1+\tilde{\alpha}_2\}<l/2$, and $\tilde{\alpha}_1+\tilde{\alpha}_2>0$ where $r_1=e^{-t}$ near $t=\infty$ and $r_2=e^t$ near $t=-\infty$ so that $\tilde{\alpha}_i=-1-\alpha_i$.

APPENDIX A: Upper and Lower Solutions and the Maximum Principle

Suppose $\bar{\Omega}=\Omega\cup\partial\Omega$ is a compact manifold with boundary, g is a nondegenerate metric on $\bar{\Omega}$, and $\tilde{K}$, $K\in C^\infty(\bar{\Omega})$. Consider

$$\Delta_g u+\tilde{K}e^{2u}=K \qquad \text{in } \Omega. \tag{A.1}$$

Recall that $u_+(u_-)$ is an *upper* (*lower*) *solution* of (A.1) if

$$\Delta_g u+\tilde{K}e^{2u}\leqslant(\geqslant)K \qquad \text{in } \Omega.$$

The following is easily established using the maximum principle.

LEMMA 1. *Suppose $\tilde{K}\leqslant 0$ in Ω, u_+ is an upper solution and u_- is a lower solution of* (A.1) *such that $u_+\geqslant u_-$ on $\partial\Omega$. Then $u_+>u_-$ in Ω, or $u_+\equiv u_-$ in Ω.*

APPENDIX B: Isolated Singularities of $\Delta u+\tilde{K}e^{2u}=0$

In [5, Sect. 17 and 18], M. Heins studied the behavior of solutions of $\Delta u=e^{2u}$ which are C^2 in $0<r=|z|<1$ (where we take $\varepsilon=1$ without loss of generality) and continuous on $|z|=1$ using a comparison with solutions of the ordinary differential equation $u''+(1/r)\,u'=e^{2u}$ which have the behavior

$$u(r)=\beta\log\frac{1}{r}+c+o(1) \qquad \text{where} \quad \beta<1 \tag{B.1}$$

or

$$u(r) = \log \frac{1}{r} - \log \left(\log \frac{1}{r} \right) + o(1) \tag{B.2}$$

as $r \to 0$. Of course, the same behavior holds for $\Delta u = Ce^{2u}$, but in this Appendix we show that if $-C_1 r^l \leqslant \tilde{K}(z) \leqslant -C_2 r^l < 0$ for $0 < r < 1$, then every C^2 solution of

$$\Delta u + \tilde{K} e^{2u} = 0 \qquad \text{in} \quad 0 < r < 1 \tag*{$(*)_0$}$$

which is continuous on $r = 1$ is of the form

$$u(z) = \alpha \log |z| + c + o(1) \qquad \text{where} \quad \alpha > -\frac{l+2}{2} \tag{B.3}$$

or

$$u(z) = \frac{l+2}{2} \log \frac{1}{|z|} - \log \left(\log \frac{1}{|z|} \right) + O(1). \tag{B.4}$$

Clearly, it suffices to consider the case $l = 0$, i.e., to establish

$$u(z) = \beta \log \frac{1}{|z|} + c + o(1) \qquad \text{where} \quad \beta < 1 \tag{B.5}$$

or

$$u(z) = \log \frac{1}{|z|} - \log \left(\log \frac{1}{|z|} \right) + O(1) \tag{B.6}$$

when

$$-C_1 \leqslant \tilde{K}(z) \leqslant -C_2 < 0. \tag{B.7}$$

(The conclusion (B.5)–(B.6) when $\tilde{K}(z) = -4$ was made in [5]; we discuss his comparison with ordinary differential equations, but then use an argument similar to that in [3] to prove the result.)

So, suppose (B.7) holds and $u(z)$ is a C^2-solution of $(*)_0$ which is continuous on $r = 1$. Let us compare u with solutions of the ordinary differential equations

$$Y'' + \frac{1}{r} Y' = C_1 e^{2Y} \tag*{$(*)_1$}$$

$$Y'' + \frac{1}{r} Y' = C_2 e^{2Y} \tag*{$(*)_2$}$$

whose solutions are of the form (B.1) or (B.2).

First we observe that $u(z)$ is dominated by a solution Y_2 of $(*)_2$. In fact, let

$$\hat{u}(r) = \max_{\theta} u(re^{i\theta})$$

and for $0<\eta<1$ let $Y_{2,\eta}(r)$ denote the solution of $(*)_2$ on $\eta<r<1$ satisfying $Y_{2,\eta}(\eta)=\hat{u}(\eta)$ and $Y_{2,\eta}(1)=\hat{u}(1)$. Then $W(z)=Y_{2,\eta}(|z|)-u(z)$ satisfies $W(z)\geqslant 0$ on $\partial\mathbf{D}$, where $\mathbf{D}=\{z\colon \eta<|z|<1\}$. Moreover,

$$\begin{aligned} 0 &= \Delta W + C_2(e^{2u}-e^{2Y}) - (C_2+\tilde{K})\,e^{2u} \\ &\geqslant \Delta W + C_2(e^{2u}-e^{2Y}) = \Delta W + c(x)\,W, \end{aligned}$$

where $Y=Y_{2,\eta}$ and

$$c(x) = C_2\,\frac{e^{2u}-e^{2Y}}{Y-u} \leqslant 0.$$

By the maximum principle, we find $W\geqslant 0$ on $\mathbf{D}$, so $Y_{2,\eta}(r)\geqslant \hat{u}(r)$ for $\eta\leqslant r\leqslant 1$. Letting $\eta\to 0$ we see that $u(z)$ is dominated by a solution Y_2 of $(*)_2$ on $0<r<1$ which must therefore satisfy (B.1) or (B.2).

Now let $Y_1(r)$ and $Y_2(r)$ denote the unique solutions of $(*)_1$ and $(*)_2$, respectively, with

$$Y_i(r) = \log\frac{1}{r} - \log\log\frac{1}{r} + o(1) \qquad \text{as} \quad r\to 0$$

$$Y_1(1) = \min\{u(z)\colon |z|=1\}, \qquad Y_2(1) = \max\{u(z)\colon |z|=1\}.$$

Then we have

$$Y_1(r) \leqslant Y_2(r) \qquad \text{for} \quad 0<r<1. \tag{B.8}$$

(This may be seen, for example, by letting $Y_\beta(r)$ for $\beta<1$ denote the solution to $(*)_1$ with $Y_\beta(1)=Y_1(1)$ and $Y_\beta(r)=\beta\log(1/r)+c+o(1)$ as $r\to 0$ so that $Y_\beta(r)<Y_2(r)$ by the maximum principle. Letting $\beta\to 1$ we obtain (B.8).)

We conclude from (B.8) that there is a solution U of $(*)_0$ satisfying

$$\begin{aligned} &U(z) = \log\frac{1}{|z|} - \log\log\frac{1}{|z|} + O(1) \qquad \text{as} \quad |z|\to 0 \\ &U(z) = u(z) \qquad \text{for} \quad |z|=1 \end{aligned} \tag{B.9}$$

In fact, $U(z)$ is the "maximal solution" of $(*)_0$ with the given boundary condition $U(z)=u(z)$ at $|z|=1$. To see this, let

$$\tilde{U}(z) = \sup\{\tilde{u}(z)\colon \tilde{u} \text{ solves } (*)_0 \text{ and } \tilde{u}=u \text{ at } |z|=1\}$$

be the maximal solution. Then $U \leqslant \tilde{U} \leqslant Y_2$ implies that $W = \tilde{U} - U$ is bounded, nonnegative, and subharmonic on $0 < |z| \leqslant 1$. Moreover, $W = 0$ at $|z| = 1$ so we can extend W by zero to a bounded subharmonic function on $0 < |z| < \infty$. Thus $W \equiv 0$ on $0 < |z| \leqslant 1$.

Thus $u(z) \leqslant U(z)$ for $0 < |z| \leqslant 1$ and we let $\varphi(z) = U(z) - u(z) \geqslant 0$. Moreover, φ is subharmonic on $0 < |z| \leqslant 1$ since $\Delta\varphi = (-\tilde{K})(e^{2U} - e^{2u}) \geqslant 0$. Hence, so is

$$\bar{\varphi}(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(re^{i\theta})\, d\theta.$$

Thus $(r\bar{\varphi}_r)_r \geqslant 0$ and we can integrate twice to obtain

$$\bar{\varphi}(r) \geqslant \bar{\varphi}_r(1) \log r + \bar{\varphi}(1). \tag{B.10}$$

There are two cases to consider: (i) $\bar{\varphi}_r(r) \geqslant 0$ for all $0 < r \leqslant 1$, or (ii) $\bar{\varphi}_r(r) < 0$ for some $0 < r \leqslant 1$.

But first, let us observe that

$$\hat{\varphi}(r) = \bar{\varphi}(r) + o(1) \tag{B.11}$$

as $r \to 0$. In fact, under the inversion $r \to 1/r$, it suffices to check (B.11) as $r \to \infty$ assuming that $\varphi(z)$ is a nonnegative subharmonic function for $|z| \geqslant 1$, satisfying $\hat{\varphi}(r) = O(\log r)$ as $r \to \infty$ and $\Delta\varphi \geqslant 0$ in $|z| > 1$ with $\Delta\varphi \in L^1(|z| > 1)$. But such a function allows a representation (cf. [6])

$$\varphi(z) = \beta \log |z| + h(z) + \frac{1}{2\pi} \int_{|\zeta| > 1} \log \left| 1 - \frac{z}{\zeta} \right| \Delta\varphi(\zeta)\, d\zeta, \tag{B.12}$$

where $h(z)$ is harmonic in $|z| > 1$ up to infinity; hence $h(z) = c + o(1)$. Now

$$\psi(z) = \frac{1}{2\pi} \int_{|\zeta| > 1} \log \left| 1 - \frac{z}{\zeta} \right| \Delta\varphi(\zeta)\, d\zeta$$

is subharmonic on $\mathbb{C}$ and satisfies the conditions of (2.10) in [4] so

$$\hat{\psi}(r) = \bar{\psi}(r) + o(1) \qquad \text{as} \quad r \to \infty.$$

Clearly this implies (B.11).

In case (i), we use $\bar{\varphi}(r) \geqslant 0$ to conclude $\bar{\varphi}(r) \to \bar{\varphi}_0 \geqslant 0$ as $r \to 0$. Then we use (B.11) to conclude $U(z) - \bar{\varphi}_0 - \delta \leqslant u(z)$ for $|z|$ small enough, which gives (B.6).

In case (ii), we may assume $\bar{\varphi}_r(1) = -\eta < 0$ so (B.10) implies $\bar{\varphi}(r) \geqslant \eta \log(1/r) + C$, hence $\bar{u}(r) \leqslant (1 - \eta) \log(1/r) + C$. Now (B.11) also holds with φ replaced by u: to see this, use the representation (B.12) for φ and the similar representation for U to represent u as in (B.12), from which (B.11) follows. Thus we find

$$\hat{u}(r) \leqslant (1-\eta) \log \frac{1}{r} + C. \tag{B.13}$$

Now we can let G denote the Green's operator on the disk $|z| \leqslant 1$ and let $v(z) = G[-\tilde{K}e^{2u}]$ which is a continuous function. Now $\Delta(u-v) = 0$ on $0 < |z| < 1$ together with (B.13) imply that $u - v$ has at most a log-singularity at $|z| = 0$, and hence that (B.5) holds.

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